But if the potential function were to have a suitable behavior to make $N(-44)=0$, the solution of the N/D equations in the strip region, $4<s<s_1$, would be very little changed because of the subtraction. Or, inverting this argument, our solution does not determine N for points a long way outside the strip with any accuracy. However, for this value of λ the $I=2$ amplitude also has a bound-state pole, as Fig. 10 shows, though again it is not possible to determine its position except that it is at $s<-300$. Since no $I=2$ trajectories are known, it seems that such solutions must be wrong. If the scattering length is to be -1.7 we require a very large $\lambda(\approx 40)$, and the $I=0$ bound-state position at $s=0.56$ is too close to the symmetry point to be identified with any known trajectory.

According to our present information the best solution is that with $\lambda=-0.1$.

ACKNOWLEDGMENTS

I would like to thank Professor G. F. Chew for many discussions and suggestions throughout the course of this work. I am also indebted to Dr. V. L. Teplitz for the use of his computer programs. The hospitality of the Lawrence Radiation Laboratory is gratefully acknowledged. The work was performed while I was the holder of a D.S.I.R. Research Fellowship.

PHYSICAL REVIEW VOLUME 142, NUMBER 4 FEBRUARY 1966

Inadequacies of the New Form of the Strip Approximation for the π-π Scattering Amplitude*

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(Received 10 September 1965)

The new form of the strip approximation is used to obtain mutually self-consistent trajectories with isopin $I=0$ and $I=1$ in the π-π system. However, these trajectories do not correspond to those which are obtained from experiment, and violate unitarity in the asymptotic region. The trajectories obtained from experiment, which satisfy unitarity, are shown not to produce sufficient strength to bootstrap themselves. Also the $I=0$ trajectory gives rise to a repulsive potential, and to obtain a solution of the N/D equations we are impelled to the doubtful assumption that this repulsion is completely cancelled by other $I=0$ trajectories that do not reach the right-half angular-momentum plane. It is concluded that both these difficulties stem from the fact that the potential is included only in the first Born approximation, and that more satisfactory results would be forthcoming if the potential were iterated in the way proposed by Mandelstam.

I. INTRODUCTION

THE new form of the strip approximation has been proposed[1,2] as a method of calculating scattering amplitudes in accordance with the principles of maximal analyticity of the first and second kinds. The amplitudes are constructed so that they satisfy the Mandelstam representation, and all their poles are Regge poles. Such amplitudes will have the correct behavior in the low-energy resonance region where the poles dominate, and also in the high-energy region where Regge asymptotic behavior is observed. It is hoped that these features include enough of the dynamics for the amplitudes to be self-consistent in the sense that the "potential" due to the crossed-channel singularities generates the direct-channel singularities.

For the π-π amplitude, in which identical processes occur in the direct and crossed channels, this self-consistency amounts to a "bootstrap" requirement. The dominant Regge trajectories, ρ, P, and P' should bootstrap themselves.

Chew and Jones[2] have devised a set of equations for investigating this possibility using the N/D method, with the N function having the cuts of the potential, and the D function the unitarity cut in the strip region. Results have already been reported[3] for a self-consistent ρ trajectory, but the ρ potential also generated an $I=0$ trajectory which was not included in the potential. In this paper we complete the solution by obtaining a pair of mutually self-consistent trajectories, one having $I=0$ and the other $I=1$. However, these trajectories have several unsatisfactory features, and we are led to discuss some deficiencies of the new form of the strip approximation, and how they might be rectified.

In the next two sections the N/D equations and the method of calculating the potential from the exchange of Regge trajectories are reviewed. The fourth section is devoted to a discussion of the potential for P exchange, which is repulsive. The total potential for $I=0$ exchange may be made attractive by means of a "normalization"

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* This work was done under the auspices of the U. S. Atomic Energy Commission.

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1 G. F. Chew, Phys. Rev. **129**, 2363 (1963).

2 G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

3 P. D. B. Collins and V. L. Teplitz, Phys. Rev. **140**, B663 (1965).

procedure[4] which is supposed to take account of the effect of trajectories which do not reach the right-half angular-momentum plane, but whose presence is implied by the fact that the elastic discontinuity must be positive. However, doubt is thrown on the validity of this normalization procedure. In Sec. V we present results for the self-consistent ρ and (normalized) P trajectories. The P' is not found. These self-consistent trajectories are unlike the physical ρ and P trajectories in several respects, having much larger residues than are found experimentally, and smaller slopes. For this reason the trajectories violate unitarity in the asymptotic region. In Sec. VI it is shown that the trajectories which are obtained from experiment, and which do satisfy asymptotic unitarity, cannot generate enough strength in the new form of the strip approximation to bootstrap themselves. In Sec. VII we discuss the inability of the N/D method to treat combinations of attractive and repulsive potentials such as we obtain if the P potential is not normalized. Both these problems seem to stem from treating the potential in the first Born approximation, and in the final section we conclude that a more adequate strength, and a better treatment of the P repulsion, would result from iterating the potential in the way originally proposed by Mandelstam.[5]

II. THE N/D EQUATIONS

The form of the N/D equations which we use has already been discussed in previous papers.[1–3] We review them here only for completeness.

We relate the partial-wave amplitude for isospin I to the total π-π scattering amplitude $A^I(s,t,u)$ by

$$A^I(s,t,u)=\sum_l (2l+1)[1+(-1)^{I+l}] \times P_l(1+t/2q_s^2)A_l^I(s). \quad \text{(II.1)}$$

We note that, because of the Bose statistics of pions, only the even signature amplitude $A^{+I}(s,t)$ exists for $I=0$, 2, and only the odd signature $A^{-I}(s,t)$ for $I=1$:

$$A^I(s,t,u)=A^I(s,t)+(-1)^I A^I(s,u). \quad \text{(II.2)}$$

The forces in the $I=2$ channel are repulsive and no trajectories are produced, so we shall limit our attention to $I=0$ and 1:

$$A_l^I(s)=e^{i\delta_l^I(s)}\sin\delta_l^I(s)/\rho(s), \quad \text{(II.3)}$$

where $\rho(s)=[(s-4)/s]^{1/2}$ is the phase-space factor, and we assume that the phase shift $\delta_l^I(s)$ is real (i.e., elastic unitarity) in the range $4<s<s_1$.

We represent the partial-wave amplitude by

$$A_l^I(s)=q_s^{2l}B_l^I(s)=q_s^{2l}N_l^I(s)/D_l^I(s), \quad \text{(II.4)}$$

where $B_l^I(s)$ is the reduced partial-wave amplitude, with the kinematical singularities at threshold removed, and $N_l^I(s)$ has the left-hand cut of $B_l^I(s)$ and its right-hand cut for $s>s_1$, and $D_l^I(s)$ has the right-hand unitarity cut from threshold to s_1. Here s, t, and u are the squares of the barycentric energies in the various channels with $q_{s,t,u}$ the corresponding momenta, and s_1 is the strip width. We use the pion mass as the unit of energy throughout.

We then obtain the equations[1]

$$N_l(s)=B_l^v(s)+\frac{1}{\pi}\int_4^{s_1} ds'\frac{B_l^v(s')-B_l^v(s)}{(s'-s)}\rho_l(s')N_l(s') \quad \text{(II.5)}$$

and

$$D_l(s)=1-\frac{1}{\pi}\int_4^{s_1} ds'\frac{\rho_l(s')N_l(s')}{(s'-s)}, \quad \text{(II.6)}$$

where

$$\rho_l(s)=[(s-4)/s]^{1/2}[(s-4)/4]^l,$$

and $B_l^v(s)$ is the partial-wave potential function.

The integral equation (II.5) is not Fredholm because, as we shall see in the next section, $B_l^v(s)$ has a logarithmic singularity at s_1. In fact

$$B_l^v(s)\underset{s\to s_1}{\longrightarrow}(1/\pi)\,\mathrm{Im}B_l^v(s_1)\ln(s_1-s), \quad \text{(II.7)}$$

and

$$\sin^2\delta_l(s_1)=\rho_l(s_1)\,\mathrm{Im}B_l^v(s_1) =\lambda_l \quad \text{(say)}. \quad \text{(II.8)}$$

This singularity serves to match the phase shift below s_1, given by the solution of the N/D equations, to the value above s_1, given by Regge asymptotic behavior. Clearly unitarity at s_1 requires that

$$\lambda_l\leqslant 1, \quad \text{(II.9)}$$

and Chew[6] has shown that if this condition is satisfied Eq. (II.5) can be transformed, by the Wiener-Hopf method, into a Fredholm equation

$$N_l^0(s)=B_l^v(s)+\int_4^{s_1} ds'\,K_l'(s,s')N_l^0(s'), \quad \text{(II.10)}$$

where $N_l^0(s)$ is related to $N_l(s)$ by

$$N_l(s)=\int_4^{s_1} ds' O_l(s,s')N_l^0(s'), \quad \text{(II.11)}$$

$O_l(s,s')$ and $K_l(s,s')$ being known[7] functions of $B_l^v(s)$, λ_l, and s_1. A FORTRAN program for solving these equations has been devised,[8] but the Wiener-Hopf trans-

[4] G. F. Chew and V. L. Teplitz, Phys. Rev. **137**, B139 (1965).
[5] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[6] G. F. Chew, Phys. Rev. **130**, 1264 (1963).
[7] V. L. Teplitz, Phys. Rev. **137**, B136 (1965).
[8] D. C. Teplitz and V. L. Teplitz, Lawrence Radiation Laboratory Report UCRL-11696, 1964 (unpublished).

formations are rather time-consuming.[9] Equation (II.10), being Fredholm, can be solved by matrix inversion.

However, more recently, Jones and Tiktopoulos[10] have shown that any integral equation, the norm of whose kernel is less than one, can be solved by matrix inversion, whether or not it is Fredholm, and that for an equation such as (II.5) this is simply the requirement that $\lambda_l<1$. Thus, if unitarity is satisfied in the asymptotic region ($s>s_1$), (II.5) can be solved, as it stands, by matrix inversion, providing that care is taken with the choice of mesh points for s near s_1.[11] In view of the difficulties in satisfying the unitarity condition we decided to use the Jones-Tiktopoulos method rather than Chew's method, which had been used in previous work.[3,9]

A pole in the partial-wave amplitude is represented by a zero of the D function, and the output trajectory is the function $\alpha(s)$ such that

$$D_{\alpha(s)}(s)=0. \qquad \text{(II.12)}$$

Above threshold both $\alpha(s)$ and $D_l(s)$ become complex, but their imaginary parts are expected to be small, in which case we can make the approximation of supposing that

$$\mathrm{Re}\{D_{\mathrm{Re}[\alpha(s)]}(s)\}=0. \qquad \text{(II.13)}$$

As previously,[3] the solutions we obtain turn out to have $\mathrm{Im}[\alpha(s)]$ large just above threshold, and, since it is much more difficult to solve the equations for complex l, we are unable to trace the trajectories above threshold.

The output residue $\gamma(s)$ is obtained from the relation[3]

$$\left[\frac{N_l(s)}{dD_l(s)/ds}\right]_{s=s_R}=\frac{\gamma(s_R)}{\alpha'(s_R)}, \qquad \text{(II.14)}$$

where s_R is the pole position.

III. THE POTENTIAL FUNCTION

In Fig. 1 we show the six regions ($i_{1,2},j_{1,2},k_{1,2}$) of the double spectral functions employed in the new form of the strip approximation.[2] The double spectral function in region j_1, for example, is given by

$$\rho(s,t)=\Delta_t[\tfrac{1}{2}\pi\Gamma_j(t)P_{\alpha_j(t)}(-1-s/2q_t^2)]\theta(s-s_1), \qquad \text{(III.1)}$$

where

$$\Gamma_j(t)=[2\alpha_j(t)+1]\gamma_j(t)(-q_t^2)^{\alpha_j(t)}, \qquad \text{(III.2)}$$

$\alpha_j(t)$ being the trajectory function, and $\gamma_j(t)$ the reduced residue function, of the jth Regge trajectory. The con-

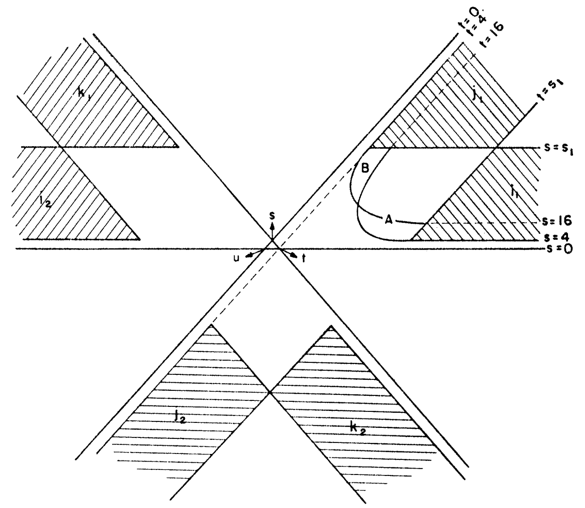

FIG. 1. The Mandelstam diagram for the new form of the strip approximation, showing the six strips (shaded) $i_{1,2}$, $j_{1,2}$, and $k_{1,2}$. The curve enclosing A is the boundary of the elastic double spectral function for the s channel, and that enclosing B is the boundary of the t-channel elastic double spectral function.

tribution of this strip to the amplitude is[2]

$$R_j^{s_1}(s_1t)=\tfrac{1}{2}\Gamma_j(t)\int_{s_1}^{\infty}\frac{P_{\alpha_j(t)}(-1-s/2q_t^2)}{(s'-s)}ds', \qquad \text{(III.3)}$$

where this integral is defined, and for $\alpha>0$ we use its analytic continuation

$$R_j^{s_1}(s,t)=\tfrac{1}{2}\Gamma_j(t)\left\{-\frac{\pi}{\sin\pi\alpha_j(t)}P_{\alpha_j(t)}\left(1+\frac{s}{2q_t^2}\right)\right.$$
$$\left.-\int_{-4q_t^2}^{s_1}\frac{P_{\alpha_j(t)}(-1-s'/2q_t^2)}{(s'-s)}ds'\right\}. \qquad \text{(III.4)}$$

The full amplitude is given by[2]

$$\begin{aligned}A^I(s,t,u)&=\textstyle\sum_i[R_i^{t_1}(s,t)+(-1)^{I_i}R_i^{u_1}(s,u)]\delta_{II_i}\\&+\textstyle\sum_j\beta(I,I_j)[R_j^{s_1}(t,s)+(-1)^{I_j}R_j^{u_1}(t,u)] \qquad \text{(III.5)}\\&+(-1)^I\textstyle\sum_k\beta(I,I_k)[R_k^{s_1}(u,s)+(-1)^{I_k}R_k^{t_1}(u,t)],\end{aligned}$$

where the sums are over the leading trajectories in the respective channels.

The reduced partial-wave amplitude for complex l is defined by

$$B_l^I(s)=-\frac{1}{2\pi}\int_{-\infty}^{0}\frac{dt}{q_s^{2l+2}}\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]A^I(s,t), \qquad \text{(III.6)}$$

a form first pointed out by Wong,[12] and the partial-wave potential function is[3] (remembering the crossing

[9] D. C. Teplitz and V. L. Teplitz, Phys. Rev. **136**, B142 (1965).

[10] C. E. Jones and G. Tiktopoulos, Princeton University report, 1965 (unpublished).

[11] I am grateful to Dr. N. Bali (Lawrence Radiation Laboratory) for discussions on this point.

[12] D. Wong, University of California, San Diego [private communication to G. F. Chew (see Ref. 1)].

symmetry)

$$B_l^{vI}(s)=\sum_j \frac{1}{2\pi q_s^{2l+2}}\int_{-\infty}^{0} dt\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]\left\{\Gamma_j^{II_j}(t)\int_{-4q_t^2}^{s_1} du'\, P_{\alpha_j(t)}\left(-1-\frac{u'}{2q_t^2}\right)\left[\frac{1}{u'-s}+\frac{(-1)^{I_j}}{u'-u}\right]\right.$$

$$+(-1)^{I_j}\int_{s_1}^{\infty} du'\,\Gamma_j^{II_j}(t')P_{\alpha_j(t')}\left(-1-\frac{u'}{2q_t^{12}}\right)\left[\frac{1}{u'-u}-(-1)^{I}\frac{1}{u'-t}\right]$$

either

$$+\frac{\pi\Gamma_j^{II_j}(t)}{\sin\pi\alpha_j(t)}\left[(-1)^{I_j}P_{\alpha_j(t)}\left(-1-\frac{s}{2q_t^2}\right)+P_{\alpha(t)}\left(1+\frac{s}{2q_t^2}\right)\right]\Bigg\}\quad \text{if}\quad \left(-1-\frac{s}{2q_t^2}\right)<1$$

or

$$+\Gamma_j^{II_j}(t)\left[\pi P_{\alpha(t)}\left(-1-\frac{s}{2q_t^2}\right)\times\begin{bmatrix}\cot(\pi\alpha_j(t)/2] & \text{for } I_j=0,\,2\\ -\tan[\pi\alpha_j(t)/2] & \text{for } I_j=1\end{bmatrix}-2Q_{\alpha_j(t)}\left(-1-\frac{s}{2q_t^2}\right)\right]\Bigg\},$$

$$\text{if}\quad\left(-1-\frac{s}{2q_t^2}\right)>1,\qquad \text{III.7}$$

$$-\frac{1}{4}\int_{-\infty}^{4-t_1} ds'\frac{\Gamma_j(s')\delta_{II_j}}{(s'-s)(-q_{s'}^2)^{l+1}}\int_{4-s'}^{t_1} dt' P_{\alpha_j(s')}\left(-1-\frac{t'}{2q_{s'}^2}\right)P_l\left(-1-\frac{t'}{2q_{s'}^2}\right),$$

where the sum is over the leading trajectories, and $s+t+u=s+t'+u'=4$. Also,

$$\Gamma_j^{II_j}(t)=\beta(I,I_j)\Gamma_j(t)\,, \qquad \text{(III.8)}$$

$\beta(I,I')$ being the isotopic-spin crossing matrix

$$\beta(I,I')=\begin{bmatrix}\frac{1}{3} & 1 & 5/3\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}. \qquad \text{(III.9)}$$

Note that

$$\begin{aligned}\mathrm{Im}[Q_l(z)]&=\tfrac{1}{2}\pi P_l(z) \quad \text{for} \quad -1<z<+1\\ &=-(\sin\pi l)Q_l(-z) \quad \text{for} \quad z<1-.\end{aligned} \qquad \text{(III.10)}$$

From the first term of (III.7),

$$\mathrm{Im}B_l^v(s_1)=\sum_j \frac{1}{2\pi q_{s_1}^{2l+2}}\int_{-\infty}^{0} dt\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^{12}}\right)\right]\pi\Gamma_j^{II_j}(t)\times P_{\alpha_j(t)}\left(-1-\frac{s_1}{2q_t^2}\right). \qquad \text{(III.11)}$$

We have included the contributions of the strips $i_{1,2}$ to the left-hand cut in $B_l^v(s)$, which we have called the partial-wave potential function. Strictly these contributions are not part of the potential, but represent the reaction to the potential. However, these extra contributions are unimportant and it seems reasonable to use the term. The "potential" is the expression in braces { } in Eq. (III.7).

IV. THE TREATMENT OF THE POMERANCHUK REPULSION

When Eq. (III.7) is used to evaluate the potential for an even signature trajectory such as the P or P' it is found that the potential function is negative (i.e., repulsive). Thus if we make the approximation of setting $\alpha=1$ we obtain

$$V^P(s,t)=\Gamma^{II_j}(t)\left\{-s_1-4q_t^2+\left(q_t^2+\frac{s}{2}\right)\times\left[\ln\left(\frac{s_1+s}{s_1-s}\right)-\ln\left(\frac{-s}{t-s-4}\right)\right]\right\}, \qquad \text{(IV.1)}$$

where $V^P(s,t)$ is the expression in braces { } in (III.7). For $t\ll s$ we get

$$V^P(s,t)\approx\Gamma^{II_j}(t)\left[-s_1+\frac{s}{s_1}\ln\left(\frac{s_1+s}{s_1-s}\right)\right], \qquad \text{(IV.2)}$$

and B_l^v can be approximated by the partial-wave projection of this expression. It will be noted that there is a repulsion depending on s_1, and the expected logarithmic singularity. For $s\ll s_1$

$$V^P(s,t)\approx\Gamma^{II_j}(t)[-s_1+s^2/s_1\cdots], \qquad \text{(IV.3)}$$

and the s^2 term is related to the spin-two (f_0) part of the P, but it is reduced by a factor s_1^2 compared with the repulsion. The lack of s dependence of the repulsion indicates that it results from the xpin-zero part of the P exchange.

Chew has shown[13] how one can understand this repulsion also in terms of the Khuri-Jones formula for $V^P(s,t)$. By expanding $V^P(s,t)$ in partial waves in the t channel, one finds

$$V^P(s,t)=\sum(2J+1)V_J(t)P_J(1+s/2q_t^2)\,, \qquad \text{(IV.4)}$$

with

$$V_J(t)=\beta(I,0)(q_t^2)^{\alpha_P(t)}\frac{\gamma_P(t)}{J-\alpha_P(t)}e^{-[J-\alpha_P(t)]\xi_1(t)}, \qquad \text{(IV.5)}$$

[13] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-16101, 1965 (unpublished); Phys. Rev. (to be published).

where

$$\xi_1(t)=\ln\{z_1(t)-[z_1^2(t)-1]^{1/2}\}$$

and

$$z_1(t)=1+s_1/2q_t^2.$$

For $t\ll s_1$ we have

$$V_2(t)=\beta(I,0)(q_t^2)\gamma_P(t)s_1^{\alpha_P(t)-2}/[2-\alpha_P(t)]$$

and (IV.6)

$$V_0(t)=-\beta(I,0)\gamma_P(t)s_1^{\alpha_P(t)}/\alpha_P(t).$$

We again see a much reduced attraction from the f_0 and a strong repulsion from spin 0.

At first sight it would seem that this repulsion must be incorrect, or at least that it must be cancelled by other contributions to $J=0$ exchange. For, suppose we use the Froissart-Gribov[2] form of the partial-wave projection instead of the Wong form (III.6), i.e.,

$$B_l(s)=\frac{1}{2\pi}\int_4^\infty \frac{dt}{q_s^{2l+2}}Q_l\left(1+\frac{t}{2q_s^2}\right)D_t(t,s), \quad \text{(IV.7)}$$

where $D_t(t,s)$ is the t-channel discontinuity of the amplitude. Then, neglecting the strips $i_{1,2}$ in Fig. 1, we have

$$B_l^v(s)=\frac{\beta(I,0)}{2\pi}\int_4^{s_1} \frac{dt}{q_s^{2l+2}}Q_l\left(1+\frac{t}{2q_s^2}\right)D_t(t,s). \quad \text{(IV.8)}$$

Because we take the double spectral functions to be zero outside the strips of Fig. 1, we can expand $D_t(t,s)$ in a convergent partial-wave series for $4<s<s_1$ and obtain

$$B_l^v(s)=\frac{\beta(I,0)}{2\pi}\int_4^{s_1} \frac{dt}{q_s^{2l+2}}Q_l\left(1+\frac{t}{2q_s^2}\right)$$
$$\times\sum_{l_t\text{ even}}(2l_t+1)\,\mathrm{Im}A_{l_t}(t)P_{l_t}\left(1+\frac{s}{2q_t^2}\right), \quad \text{(IV.9)}$$

and since, for elastic unitarity, $\mathrm{Im}A_{l_t}(t)$ must be positive, we can see that $B_l^v(s)$ must be positive. Thus, if the strip approximation is to be correct, there must be other contributions to $\mathrm{Im}A_{l_t}(t)$ apart from the P (or P') trajectory. These could be provided by trajectories which do not reach the right-half angular-momentum plane, but give a background contribution. (See Fig. 2.) Since such trajectories are not manifest, either physically, or in this type of calculation, there can be no hope of including them individually in the bootstrap scheme, but Chew and Teplitz[14] have shown how to represent their effect by "normalizing" the potential. This procedure consists simply of subtracting from $V^P(s,t)$ the part $V^P(0,t)$ and then adding back $V^P(0,t)$ from (IV.4).

[14] G. F. Chew and V. L. Teplitz, Phys. Rev. **137**, B139 (1965).

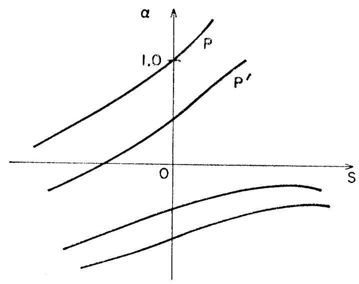

FIG. 2. The $I=0$ trajectories, including two hypothetical low-lying trajectories which might give a positive background contribution to the potential.

Thus

$$B_l^v(s)$$
$$=\frac{1}{2\pi q_s^{2l+2}}\int_{-\infty}^0 dt\left[\mathrm{Im}Q_l\left(1+\frac{t}{2q_s^2}\right)\right]\{V^P(s,t)-V^P(0,t)\}$$
$$+\frac{\beta(I,0)}{2\pi q_s^{2l+2}}\int_4^{s_1} dt\,Q_l\left(1+\frac{t}{2q_s^2}\right)\sum_{l_t\text{ even}}(2l_t+1)$$
$$\times\mathrm{Im}A_{lt}(t), \quad \text{(IV.10)}$$

and $\mathrm{Im}A_{l_t}(t)$ can be made self-consistent in the s and t channels. It turns out that the second term of this equation is very small, and in practice we shall simply use the first term, which is itself sufficient to ensure that the $I=0$ exchange force is positive. It will be seen in Sec. VI that this "normalization" drastically alters the form of the potential function. It would be difficult to add the second term in a self-consistent way because an adequate solution for $l=0$, which is sensitive to short-range forces not included in the strip approximation, cannot be found, and our solutions for $l=2$ and higher cannot be believed because we have considered only a single two-body channel whose particles (pions) have no spin. Our conclusion that this term would be small if it were included is based on the fact that even the force from saturated unitarity in the S wave $[\mathrm{Im}A_0(t)=1/\rho_0(t)]$ is small; and for the D wave, the contribution of a fixed-spin f_0, when modified by Chew's form factor,[13] is negligible.

However, the argument presented to show that the $I=0$ force must be positive could well be incorrect, since it assumes that elastic unitarity holds for $4<t<s_1$. In fact it will hold exactly only for $4<t<16$. The double spectral functions are really nonzero within the boundaries shown in Fig. 1. The region A contains the elastic double spectral function for the s channel, which, by definition, should not be included in the potential, but which will contribute to the t-channel discontinuity where it represents inelastic processes. This sort of contradiction between the sign of the t-channel discontinuity and the s-channel potential has been noted previously,[15] and we shall consider it further in the final section.

In the next section we describe the results obtained when the normalization procedure, whatever its merits, is in fact used.

[15] P. D. B. Collins, Phys. Rev. **139**, B696 (1965).

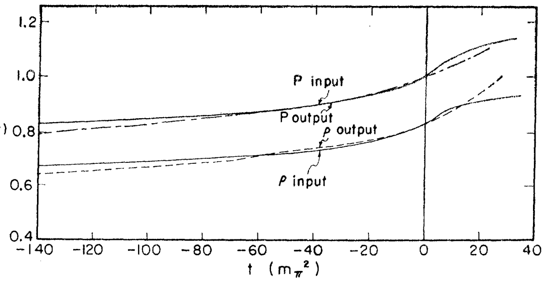

FIG. 3. A comparison of the input and output, approximately self-consistent ρ and P trajectories. The input trajectories are: $\alpha_\rho = 0.55 + 0.27/(1-t/70), \alpha_P = 0.625 + 0.375/(1-t/110)$.

V. THE BOOTSTRAP TRAJECTORIES

In a previous paper[3] a self-consistent ρ trajectory was obtained, but the ρ force also produced an $I=0$ trajectory which had not been included in the input. If now we include a normalized Pomeranchuk trajectory it is possible to obtain a completely self-consistent solution.

As in Ref. 3 we use a pole formula to parametrize the trajectory functions, insisting that the $I=1$ trajectory pass through $\alpha=1$ for $t=28$, corresponding to the ρ particle, and that the P pass through $\alpha=2$ for $t=80$, corresponding to the f_0, and through the unitarity limit $\alpha=1$ for $t=0$.

Thus

$$\alpha_\rho(t) = 1 - at_B/28 - a(1-t_B/28)/(1-t/t_B), \quad \text{(V.1)}$$

where $(1-a)$ is the intercept of the trajectory with $t=0$, and

$$\alpha_P(t) = 2 - t_A/80 - (1-t_A/80)/(1-t/t_A), \quad \text{(V.2)}$$

where t_A and t_B are the positions of the poles, which we expect to lie towards the upper end of the strip.

The same type of parametrization of the residue as was used in Ref. 3, making use of the Chew-Teplitz formula, was found to be satisfactory, i.e.,

$$\gamma_\rho(t) = C_\rho \alpha_\rho'(t)[\bar{t}_\rho - t] Q_{\alpha_\rho(t)}\left(1 + \frac{56}{\bar{t}_\rho - 4}\right) \Big/ \left(\frac{\bar{t}_\rho - 4}{4}\right)^{\alpha_\rho(t)+1}, \quad \text{(V.3)}$$

$$\gamma_P(t) = C_P \alpha_P'(t)[\bar{t}_P - t] Q_{\alpha_P(t)}\left(1 + \frac{56}{\bar{t}_P - 4}\right) \Big/ \left(\frac{\bar{t}_P - 4}{4}\right)^{\alpha_P(t)+1}. \quad \text{(V.4)}$$

The merit of this parametrization stems from the fact that the principal force in the system is ρ exchange, and the Reggeized ρ force is similar in its energy dependence to the elementary ρ force. The parameter $\bar{t}_{\rho,P}$ is some mean energy within the strip. Thus in searching for self-consistency we have to vary the Regge parameters a, t_A, t_B, C_ρ, C_P, $\bar{t}_\rho$, and $\bar{t}_P$, and the strip width s_1.

For a given choice of the parameters we calculate the potential, using (III.7) for a range of l and s, and then solve the N/D equations for $I=0$ and 1, obtaining output residue and trajectory functions. It was indicated previously[3] that a large amount of computer time is required to find self-consistent solutions, but the self-consistency is now much more nearly unique than it was for the ρ alone. Only with $80 < s_1 < 130$ is one able to approach self-consistency at all closely, and we concentrated on $s_1 = 100$. We were then able to obtain fairly good self-consistency for $a = 0.18$, $t_A = 110$, $t_B = 70$, $C_\rho = 125$, $C_P = 230$, $\bar{t}_\rho = 0.40$, and $\bar{t}_P = 50$.

A comparison of the input and output trajectory and residue functions is given in Figs. 3 and 4. A slightly different parametrization of the γ's might have resulted in some improvement, but the solution shown is obviously close to the best self-consistent solution.

Ideally, in a bootstrap calculation, one ought not to fix any of the parameters beforehand, but, since the force only depends on $\alpha(t)$ for $t<0$, fixing $\alpha_\rho(28)=1$ is not a strong restriction. If the trajectory is made to lie too high or too low, the force obtained is too strong or too weak for self-consistency to be possible. Fixing $\alpha_P(0)=1$ does, however, restrict the solution greatly. The self-consistent ρ alone reported in Ref. 3 gave rise to a P trajectory which exceeded the unitarity limit. It is the exclusion of this type of solution which has caused the greater restriction in the range of the parameters for which approximate self-consistency can be obtained.

As before,[3] the output diverges widely from the input as t becomes positive, the rapid variation of the output α's and γ's indicating that they have large imaginary parts just above threshold. The input parameters $\gamma_\rho(28)/\alpha_\rho'(28)$ correspond to a ρ width of $1.2m_\pi$, but the output $I=1, l=1$ cross section (Fig. 5) shows a

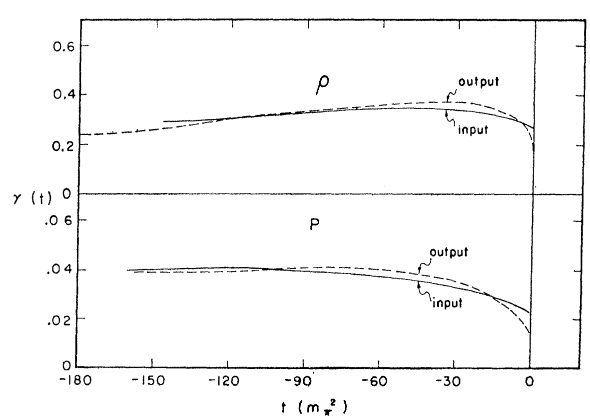

FIG. 4. A comparison of the input and output, approximately self-consistent ρ and P trajectories. The input residue functions are:

$$\gamma_\rho = 125\alpha_\rho'(t)(40-t)Q_{\alpha_\rho(t)}(2.55)/(9)^{\alpha_\rho(t)+1},$$

$$\gamma_P = 230\alpha_P'(t)(50-t)Q_{\alpha_\rho(t)}(2.22)/(11.5)^{\alpha_P(t)+1}.$$

width of $4.2m_\pi$. In fact, the equivalent input ρ width is much larger than $1.2m_\pi$ because of the unrealistic way in which our residue function decreases for $t>0$. Similarly the input $\gamma_\rho(0)$ corresponds[16] to a π-π total cross section of 36 mb rather than the expected 11 mb deduced from Ref. 17. As in Ref. 3, it was not possible to find solutions in which $\gamma(t)$ falls off sharply as t is decreased from zero, and it was thought wise to set $\gamma(t)=0$ artificially for $t<-s_1+4$. This makes the second term of (III.7) zero, whereas it would be much larger than is consistent with the strip approximation were this cutoff not imposed. As was mentioned in Ref. 3, it is necessary to have rapidly decreasing residues in order to produce steep trajectories, and it is the failure to obtain such residues as output which forces us to solutions with high trajectories, quite unlike those found from experiment in Ref. 18.

In fact, our trajectories are not admissible solutions to the problem because they do not satisfy unitarity in the asymptotic region. From Eq. (II.8), we see that for unitarity to be satisfied at s_1 we require

$$\lambda_l=\rho_l(s_1)\ \mathrm{Im}B_l^V(s_1)\leqslant 1. \qquad \text{(V.5)}$$

A plot of λ_l versus l is shown in Fig. 6, where we see that this condition is not satisfied for $l<0.82$ for $I=1$, and $l<0.95$ for $I=0$. The difference between the two isotopic spins is simply that the ρ trajectory contributes twice as strongly to $I=0$ because of the crossing matrix. As we mentioned in Sec. II, $\lambda_l<1$ is the condition for matrix inversion to give the solution of the integral equation (II.5), and so the trajectories plotted in Fig. 3 are not to be relied upon below these values of l. However, there is no discontinuity in the solution of the matrix inversion equations as λ_l becomes greater than one, so we can expect the solution obtained to be close to that which would be obtained if unitarity were not

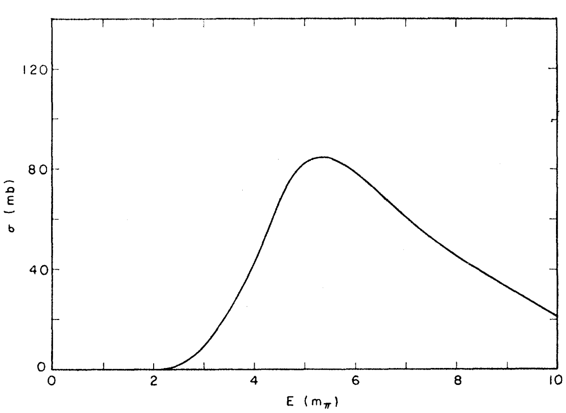

FIG. 5. The $I=1, l=1$ cross section obtained with the self-consistent trajectories. The width is about $4.2m_\pi$.

[16] See G. F. Chew and V. L. Teplitz, Phys. Rev. **136**, B1154 (1964).
[17] R. J. N. Phillips and W. Rarita, Phys. Rev. Letters **14**, 502 (1965).
[18] R. J. N. Philips and W. Rarita, Phys. Rev. **139**, B1336 (1965).

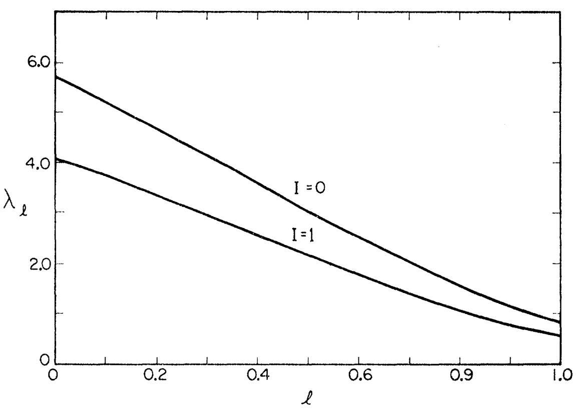

FIG. 6. A plot of λ_l versus l for the self-consistent trajectories, showing that the unitarity condition, $\lambda_l\leqslant 1$, is satisfied only for the tops of the trajectories.

violated. But Fig. 6 shows that for $l=0$ unitarity is so far from being satisfied that the self-consistent trajectories we have found cannot possibly correspond at all closely to the physical trajectories. One can see from Eq. (III.11) that if α and γ did decrease more rapidly, and γ were smaller, this problem would not arise. In fact, Phillips and Rarita[18] checked that their trajectories satisfied unitarity at high energy.

Our conclusion from the results reported in this section is that there is no bootstrap solution to the Chew-Jones equations which satisfies unitarity in the asymptotic region, but that, if we ignore this condition, trajectories which are completely self-consistent for $t<0$ can be found, which bear a rather remote resemblance to those determined from experiment.

There was no sign of a secondary $I=0$ trajectory corresponding to the P', even when attempts were made to include it as a force.

VI. FURTHER ASPECTS OF THE REGGE POTENTIALS

Despite our inability to find a bootstrap solution of the Chew-Jones equations it is important to understand the nature of the potentials produced by Regge poles.

Chew has shown[13] how the fixed-spin exchange potential is modified, by what he calls a "form factor," when continuation in angular momentum is taken into account. This form factor may enhance or reduce the force, depending on the spin of the particle exchanged. For a particle of high spin the form factor always results in a reduction of the force, but for the ρ the situation is less certain.

It is well known that though the ρ is the principal force in the π-π problem, a fixed-spin particle of the physical width does not give a sufficient strength to bootstrap itself in the first Born approximation.

The potential function is

$$B_l^\rho(s)=3gm_\rho\left(1+\frac{2s}{m_\rho^2-4}\right)Q_l\left(1+\frac{m_\rho^2}{2q_l^2}\right), \qquad \text{(VI.1)}$$

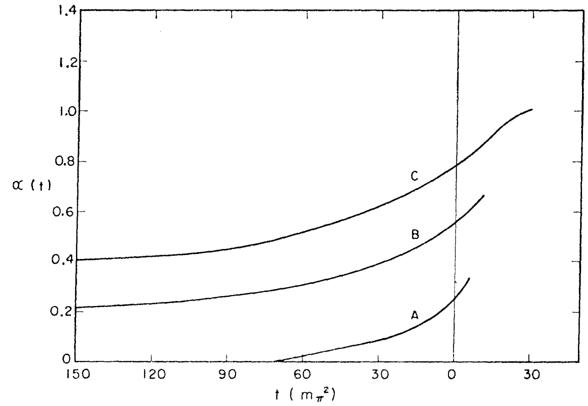

FIG. 7. The trajectories obtained with the exchange of a fixed-spin ρ of width—$A=0.7m_\pi$, $B=1.4m_\pi$, and $C=2.3m_\pi$, with a cutoff at $s_1=200m_\pi^2$.

where g is the width of the exchanged ρ in pion mass units, and the results of using such a force in the N/D equations with a cutoff at $s_1=200$ are shown in Fig. 7. It is seen that a width of $2.3m_\pi$ is required to produce a trajectory which passes through m_ρ^2 at $\alpha=1$. We compare this with the force obtained from two different parametrizations of the ρ trajectory in Fig. 8. The ρ of case (b) produces a force which is similar to that from the fixed-spin ρ for $l=1$, though it is smaller for lower values of l, but it still suffers from the difficulty that unitarity in the asymptotic region cannot be satisfied

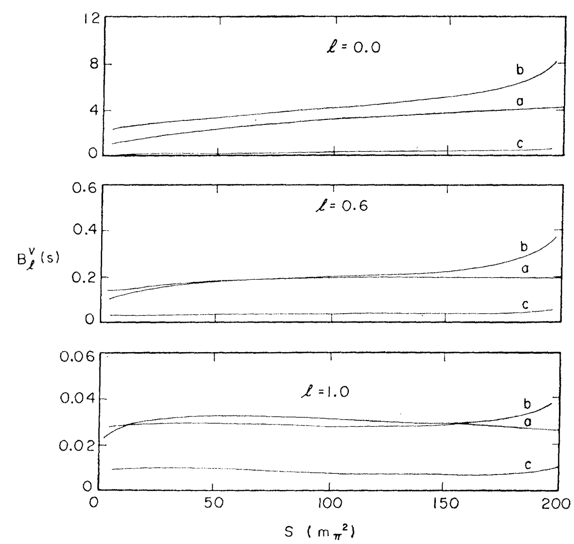

FIG. 8. A comparison of the potential functions, $B_l^v(s)$, for three values of l, with $s_1=200m_\pi^2$. The input parameters for the three cases are:

(a) Fixed spin ρ of width $0.7m_\pi$.
(b) A ρ trajectory with the parameters:
$\alpha_\rho=0.107+0.393/(1-t/50)$; $\gamma_\rho=0.22\times(49)^{1-\alpha_\rho(t)}/(1-t/200)$.
(c) A ρ trajectory with the parameters:
$\alpha_\rho=-1.5+2/(1-t/140)$; $\gamma_\rho=0.01\times(24)^{1-\alpha_\rho(t)}/(1-t/100)$.

Both sets of parameters correspond to a physical ρ of width $0.7m$. [The presence of (numerical factor)$^{-\alpha(t)}$ is to make $\Gamma(t)\propto\gamma(t)\times(-q_t^2)^{\alpha(t)}$ dimensionless.]

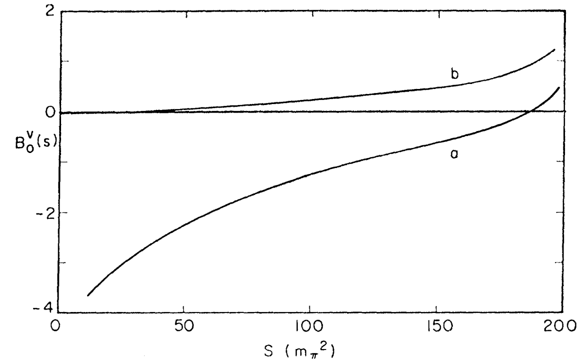

FIG. 9. A comparison of the unnormalized (a), and normalized (b), potential functions for an input P trajectory having the parameters:

$$\alpha_P=-1.0+2.0/(1-t/240);$$
$$\gamma_P=0.007\times(24)^{1-\alpha_P(t)}\times\alpha_P(t)/(1-t/100)$$

for $l=0$. These parameters correspond to a π-π total cross section of 1 mb. The multiplication of the residue by $\alpha(t)$ is to ensure that it vanishes where the trajectory cuts angular momentum zero, so that there is no "ghost" pole.

even if we neglect the fact that the P and P' trajectories also contribute strongly to $\mathrm{Im}B_l^v(s)$. In case (c) the input trajectories are similar to those found by Phillips and Rarita,[18] and the force is much smaller.

The addition of the even-signature trajectories is of little help because if the normalization procedure is used the remaining P force is very small, as Fig. 9 shows. If we use input parameters based on those of Phillips and Rarita for the ρ and (normalized) P the force is too weak to produce any output trajectory at all. If the P contribution is not normalized the resulting total force is repulsive, and it is not possible to obtain a sensible solution to the N/D equations. This point is taken up in the next section.

To summarize, the forces from the sort of trajectories found by Phillips and Rarita are even smaller than those from the exchange of a fixed-spin ρ with the experimental width, and are too weak to produce any output trajectories. The force can be increased by using residue and trajectory functions which fall less rapidly with increasing $|t|$, but if they are flattened sufficiently to produce a force equal to that from the exchange of a fixed-spin ρ, unitarity in the asymptotic region is violated at least for low angular momentum, and even this force is too weak by a sizeable factor to produce output trajectories corresponding to experiment.

VII. THE REPULSION AND THE N/D METHOD

We can see from Figs. 8 and 9 that, if we do not normalize the P contribution, the P repulsion is much greater than the ρ attraction for both the $I=0$ and $I=1$ channels except for s near s_1. If we try to solve the N/D equations with such forces we obtain trajectories of poles with negative residues, that is to say we find "ghost" resonances which lie on the physical sheet (at a in Fig. 10). What is more, the stronger the repulsion the more highly bound these resonances become. Since dynamical calculations have usually obtained resonance widths which are too large[19] (e.g., at b in Fig. 10) it might be hoped that some more moderate amount of repulsion would result in narrow resonances (at c).

To explore this phenomenon further we examined a potential scattering model in which there was a similar combination of attractive and repulsive potentials, i.e.,

$$V(r)=-g_1(e^{-m_1 r}/r)+g_2(e^{-m_2 r}/r),\quad m_1<m_2. \qquad \text{(VII.1)}$$

If one solves the Schrödinger equation with any such potential one is guaranteed that any resonances produced will lie on the unphysical sheet, but if we use the first Born approximation

$$B_l^v(s)=-\frac{g_1}{2q_s^{2l+2}}Q_l\left(1+\frac{m_1}{2q_s^2}\right)+\frac{g_2}{2q_s^{2l+2}}Q_l\left(1+\frac{m_2}{2q_s^2}\right) \qquad \text{(VII.2)}$$

and solve the N/D equations (II.5), (II.6) with a nonrelativistic phase-space factor

$$\rho_l(s)=(s-4/4)^{l+1/2} \qquad \text{(VII.3)}$$

and let $s_1\to\infty$, we find similar ghost trajectories. For example, at $l=0$, with $g_2=10.5$, $m_2=10$, and $m_1=1$ (there is no particular reason for these values), we find that for $g_1=0$ a normal bound state is produced at $s=-3.1$ and $N_0\approx B_0^v$. If we add repulsion by increasing g_1 there is, paradoxically, greater attraction, so that for $g_1=2.6$ the bound state has moved to -23. Plots of B_0^v, N_0, and D_0 are given in Fig. 11(a). For $g_1=2.7$ a pole is produced in the N function at threshold and the bound state moves to $-\infty$. For $g_1=2.8$ the N function just above threshold has changed sign [Fig. 11(c)], with a ghost pole appearing at $s=8.5$. We see that again $N_0\approx B_0^v$, because the changing sign of N makes the contribution of the integral in (II.5) very small, but this clearly does not mean that the first Born approximation still holds good. Increasing g_1 further increases the binding of the ghost resonance.

This result stems from the failure of the first Born approximation, and would be improved if we were to iterate the potential in the way suggested by Mandelstam.[5] After an infinite number of iterations the solution to the Schrödinger equation would be obtained,[20] and it would not be possible to have resonances except on the unphysical sheet. One would expect the iterations to be more important for a repulsive potential than for an attractive one because of the alternating signs of the successive iterations in the former case.

[19] This fact is discussed by, e.g., J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. **137**, B1242 (1965).

[20] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

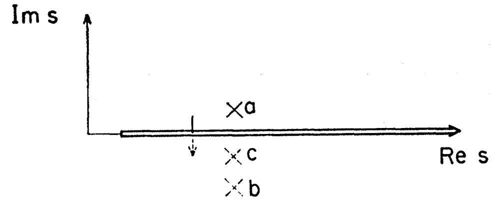

FIG. 10. Positions of resonance poles in the complex s plane at: (a) on the physical sheet, (b) well onto the unphysical sheet, giving a wide resonance, (c) just onto the unphysical sheet, giving a narrow resonance.

It would seem that in our relativistic problem we are facing the same sort of inadequacy of the Born approximation.

The normalization procedure is a valid way of correcting the t-channel discontinuity in the range $4<t<16$ where elastic unitarity is exact, and probably further out than this, since we expect the discontinuity of the potential to be small in the whole of the lower part of the strip where there are narrow resonances. This is because the double spectral function in the strip is approximately proportional to $\text{Im}\alpha(t)$ [see Eq. (III.1)] and the width of a resonance at t_R is[21]

$$\Gamma=\text{Im}\alpha(t_R)\Big/2t_R^{1/2}\left(\frac{d\,\text{Re}\alpha}{dt}\right)_{t_R}. \qquad \text{(VII.4)}$$

In the upper region of the strip we can have no such confidence in the normalization procedure, since here the discontinuity has contributions not only from the strips but also from the corner section of the double spectral function (A in Fig. 1). This corner is not included in the potential, by definition, but if it is an important contribution to the amplitude its neglect is

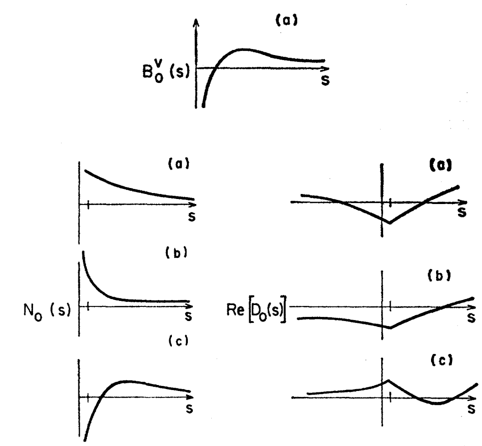

FIG. 11. A comparison of the potential function, $B_l^v(s)$, and the N and D functions, for the potential model described in the text. The three cases differ in having (a) $g_1=2.6$, (b) $g_1=2.7$, and (c) $g_1=2.8$.

[21] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

a serious defect of the new form of the strip approximation.

This part of the double spectral function could be calculated by means of the Mandelstam iteration procedure, from the equations

$$\rho(s,t)=\frac{1}{\pi q_s\sqrt{s}}\int\int_{K=0}^{\infty} dt'dt''\frac{D_t(t',s_+)D_t(t'',s_-)}{K^{1/2}(s,t,t',t'')}, \quad \text{(VII.5)}$$

where $D_t(t,s)$ is the t-channel discontinuity, and

$$K(s,t,t',t'')=\left[t^2+t'^2+t''^2-2(tt'+t't''+t''t)-\frac{tt't''}{q_s^2}\right]$$

and

$$D_t(t,s_\pm)=V_t(t,s)+\frac{P}{\pi}\int ds'\frac{\rho(s',t)}{s'-s}\pm i\rho(s,t). \quad \text{(VII.6)}$$

Starting with the t discontinuity of the potential, $V_t(t,s)$, one could calculate $\rho_A(s,t)$, the elastic s double spectral function in the region A [though this would require a knowledge of the residue and trajectory functions above threshold where they are complex, a region which hitherto we have been able to avoid by using the Wong partial-wave projection (III.6)]. If we knew $\rho_A(s,t)$ we could find its contribution to $B_l^v(s)$, and this would enable us to treat the P force properly instead of using the normalization procedure for $t\gg 16$, where its validity is rather doubtful.

However, if $\rho_A(s,t)$ is important, so by symmetry is $\rho_B(s,t)$, the elastic t double spectral function obtained by iterating the s-channel discontinuity, and this implies that the assumption of elastic unitarity in the s strip for the N/D equations is incorrect; $\rho_B(s,t)$ will also contribute to $B_l^v(s)$, of course.

Thus the new form of the strip approximation is seen to have two deficiencies, in that the cuts of the N function are taken to be simply those of the potential, and elastic unitarity is supposed to hold right out to the boundary of the strip.

Chew has argued[22] that a proper inclusion of the P repulsion should result in a narrowing of the output resonances. The argument is based partly on the fact that, in controlling the asymptotic behavior, the P is the main contribution to the potential for $s>s_1$, and represents the effect of the many channels opening up above the resonance region, and in classical nuclear physics it is the presence of such channels which is responsible for the narrow resonances. In terms of the N/D equations, the inclusion of the long-range P repulsion should reduce the N function near threshold, and hence the width of low-energy resonances, without greatly altering the position of the zero of the D function, which depends on the shorter-range ρ force.

[22] G. F. Chew, Lawrence Radiation Laboratory Report UCRL-16245, 1965 (unpublished); Phys. Rev. (to be published).

VIII. CONCLUSIONS

We have not succeeded in bootstrapping trajectories in the new form of the strip approximation as it stands. This appears to be due to the treatment of the potential in the first Born approximation, which does not produce sufficient force to regenerate the true physical trajectories when the physical trajectories are used as the input, and gives rise to a repulsion from the P and P' trajectories which we are only able to cope with by making doubtful assumptions about the presence of a background contribution. If we neglect the requirement that the input forces should correspond to the known trajectory parameters, we find that it is possible to obtain self-consistent trajectories, but these violate unitarity very seriously in the asymptotic region, require the input of a ρ resonance of too large a width, and result in an even larger output width.

It is hoped that by iterating the potential it will be possible to include the P force properly and obtain narrower resonances, and that the iteration will produce sufficient extra strength from both ρ and P to make up the deficit. However, in view of the fact that it will no longer be possible to identify the left- and far right-hand cuts of the partial-wave amplitudes with those of the potential, nor to use elastic unitarity within the strip, there will be difficulties in using the N/D method. Rather one might try to obtain crossing analyticity by iterating the potential from a given set of trajectories out to the asymptotic region, and discover whether the asymptotic behavior appears to be controlled by identical trajectories in the crossed channel. The success of Bransden *et al.*[23] in iterating a non-Regge potential and obtaining sensible output trajectories gives strong grounds for hoping that this approach will succeed, and it is expected that results will be available before very long.[24]

ACKNOWLEDGMENTS

I am indebted to Professor G. F. Chew for innumerable discussions and suggestions throughout the course of this work. I am also grateful to Dr. V. L. Teplitz, whose computer programs made these calculations possible, and to Dr. N. Bali for enlightening conversations. The hospitality of the Lawrence Radiation Laboratory is gratefully acknowledged. The work was performed while I was the holder of a D.S.I.R. Research Fellowship.

[23] B. H. Bransden, P. G. Burke, J. W. Moffat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **30**, 207 (1963).

[24] N. Bali, Lawrence Radiation Laboratory (private communication).